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On a scale function for testing the conformality of a Finsler manifold to a Berwald manifold

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Abstract

In this paper, we are going to discuss the problem whether how we can check the conformality of a Finsler manifold to a Berwald manifold. The method is based on a differential 1-form constructing on the underlying manifold by the help of integral formulas such that its exterior derivative is conformally invariant. If the Finsler manifold is conformal to a Berwald manifold, then the exterior derivative vanishes. This gives the following necessary condition: the differential form is closed and, at least locally, it is exact as the exterior derivative of a scale function for testing the conformality. A necessary and sufficient condition is also given in terms of a distinguished linear connection on the underlying manifold – it is expressed by the help of canonical data. In order to illustrate how we can simplify the process in special cases Randers manifolds are considered with some explicit calculations.

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Introduction

It is well known due to Hashiguchi and Ichijyō that a Finsler manifold is conformal to a Berwald manifold if and only if it is a Wagner manifold [9] (see also [22,24]). The Wagner manifolds form an important class of the so-called generalized Berwald manifolds admitting Finsler connections whose horizontal part depends only on the position – more precisely there exists a linear connection on the underlying manifold such that the indicatrix hypersurfaces are invariant under the parallel transport. Berwald manifolds in the classical sense are characterized by a similar property of the canonical Berwald connection. If it has zero (horizontal) curvature, then the Berwald manifold is called a locally Minkowski manifold.

For the *two-dimensional conformality problem*, the original result due to Wagner [29] states that *a Finsler space of dimension 2 is a generalized Berwald space if and only if the first derivative of the main scalar by the Landsberg angle gives a differential equation of the form*

$$y' = f(y) \tag{1}$$

(see also [13]). It is obviously a necessary condition for a two-dimensional Finsler manifold to be conformal to a Berwald manifold. Further important results are in Kikuchi's paper [11]. *He found a conformally invariant Finsler connection for all Finsler spaces satisfying a certain condition by an excellent idea and the conformal flatness was stated in terms of this connection* as Matsumoto wrote in his paper [16]. The conformal flatness means that the manifold is conformal to a locally Minkowski manifold. Note that a positive definite two-dimensional Berwald manifold must be a locally Minkowski manifold unless it is Riemannian (see Szabó's observation [18] and [10]). Indeed, in case of dimension 2 Kikuchi's results are equivalent to the Matsumoto's equations [15] (see also [16]). They consist of necessary and sufficient conditions for a two-dimensional Finsler manifold *with non-zero* J to be conformal to a Berwald manifold. The Matsumoto's equations are also differential equations in terms of the standard two-dimensional setting such as the main scalar and its derivatives with respect to special vector fields constituting the so-called Berwald frame (for the original version, see [3,4]). Under the notations of Matsumoto's paper, the function J is just the derivative of the main scalar along the parameterized indicatrix curve. In other words, the condition provides that the right-hand side of the Wagner's differential equation (1) is nontrivial. According to this additional property, the system of partial differential equations for the scale function involves J as a divisor although there always exist critical parameters because of the periodicity of the main scalar along the indicatrix.

In case of the multidimensional Kikuchi's condition, the problem is essentially similar: *Now we assume* $\det W_k^r \neq 0$, *where*

$$W_k^r = \frac{\partial}{\partial y^i}(L^2C^2)B_k^{ir}.$$

Then *there exists* W_j^{*k} *such that*

$$W_k^r W_j^{*k} = \delta_j^r$$

(for the citation, see [11]). Unfortunately, the function L^2C^2 is homogeneous of degree zero. This means that it attains both its (global) maximum and minimum on the tangent spaces and all of the partial derivatives with respect to the variables $y^1, \ldots, y^n$ vanish at such an extremal point.

In what follows, we are going to discuss not only the problem of conformal flatness but also the *multidimensional conformality problem* of Finsler manifolds to Berwald manifolds. The theory will be presented in terms of conformally invariant differential forms constructing on the *underlying manifold* by the help of integral formulas – the only requirement is that we have a non-Riemannian Finsler manifold under consideration.

1. Preliminaries

1.1. Finsler manifolds

Let M be a connected differentiable manifold equipped with a function $E : TM \to \mathbb{R}$ such that

(i) $\forall v \in TM \setminus \{0\} : E(v) > 0$ and $E(0) = 0$.
(ii) E is homogeneous of degree 2, i.e. $\forall t \in \mathbb{R}^+ : E(tv) = t^2 E(v)$.
(iii) E is of class C^1 on the tangent manifold TM and smooth except the zero section.
(iv) The *fundamental form* $\omega := dd_J E$ is nondegenerate.

The *Riemann–Finsler metric* of the Finsler manifold (M, E) is defined by the formula

$$g(JX, JY) := \omega(JX, Y),$$

where X, Y are vector fields on TM and J is the canonical almost tangent structure on the tangent bundle $\pi : TM \to M$ (for the details, see [5,6,20]). The Finsler manifold is called *positive definite* if g is positive definite.

Remark 1. In what follows, we suppose that the Finsler manifold is positive definite without any further comment.

Note that for any point $p \in M$ the restriction $g_p := g|_{\mathcal{T}_pM}$ is a Riemannian metric on the "manifold" $\mathcal{T}_pM := T_pM \setminus \{0\}$ in the usual sense. The indicatrix hypersurface at the point p is defined as follows:

$$S_p := \{v \in T_pM | L(v) = 1, \text{ where } E = \tfrac{1}{2}L^2\}.$$

1.2. The gradient operator

Let a smooth function $\varphi : TM \to \mathbb{R}$ be given. Since the fundamental form ω is nondegenerate, there exists a unique vector field grad φ such that

$$\iota_{\operatorname{grad} \varphi}\omega = \mathrm{d}\varphi.$$

This vector field is called the gradient of φ. Note that the gradient vector field is smooth only on the splitted tangent manifold

$$\mathcal{T}M := TM \setminus \{0\}.$$

In general, *differentiability is guaranteed over* $\mathcal{T}M$ unless otherwise stated.

Lemma 1 ([19]). *Let α be a smooth function on the underlying manifold, then*

$$\operatorname{grad} \alpha^{\mathrm{v}} E = \alpha^{\mathrm{c}},$$

where $\alpha^{\mathrm{v}} := \alpha \circ \pi$ is the vertical lift and

$$\alpha^{\mathrm{c}} : TM \to \mathbb{R}, \quad \alpha^{\mathrm{c}}(v) := v(\alpha)$$

is the complete lift of the function α.

1.3. Further formulas [5,6,20]

Let h be the canonical horizontal endomorphism (the so-called *Barthel endomorphism*) associated with the *canonical spray* $S := -\operatorname{grad} E$, we have

$$\iota_S\omega = -\mathrm{d}E, \qquad h := \tfrac{1}{2}([J, S] + 1).$$

Using the prolongation

$$g_{\mathrm{h}}(X, Y) := g(JX, JY) + g(\nu X, \nu Y), \quad \nu := 1 - h$$

of the Riemann–Finsler metric along h the *lowered first Cartan tensor* of the Finsler manifold is defined by the formula

$$\mathcal{C}_\flat(X, Y, Z) := \tfrac{1}{2}(\mathcal{L}_{JX} J^* g_{\mathrm{h}})(Y, Z),$$

where

$$J^* g_{\mathrm{h}}(X, Y) := g(JX, JY),$$

and $\mathcal{L}$ denotes the Lie-derivative (with respect to JX) as usual. The Barthel endomorphism determines an almost complex structure F on the splitted tangent manifold $\mathcal{T}M$ such that

$$F \circ J = h, \qquad F \circ h = -J.$$

Using the standard technical tools of tangent bundle differential geometry such as the vertical and complete lifts X^{v} and X^{c} of a vector field $X \in \mathfrak{X}(M)$ we define the horizontal lift X^{h} as follows:

$$X^{\mathrm{h}} := h(X^{\mathrm{c}}) \Rightarrow FX^{\mathrm{v}} = X^{\mathrm{h}}, \quad FX^{\mathrm{h}} = -X^{\mathrm{v}}$$

(see, e.g. [12,28]). As it is well known, h induces a (in general) nonlinear covariant derivative operator ∇ on the underlying manifold:

$$\nabla_X Y := \kappa \circ TY \circ X,$$

where κ is the connection map of h.

Definition 1. If the induced covariant derivative operator is linear, then the Finsler manifold is called a *Berwald manifold*.

1.4. The associated Riemannian metric

Suppose that the manifold M is orientable and consider a volume form $\eta \in \wedge^n(M)$. Then for any point $p \in M$ we have an orientation represented by η_p on the tangent space T_pM. Let us define the mapping

$$\mathrm{d}\mu : p \in M \to \mathrm{d}\mu_p \in \wedge^n(T_pM)$$

as follows:

$$\mathrm{d}\mu_p(X_1^{\mathrm{v}}, \dots, X_n^{\mathrm{v}}) := \begin{cases} \sqrt{\det\, g(X_i^{\mathrm{v}}, X_j^{\mathrm{v}})} & \text{if } \eta(X_1, \dots, X_n)(p) > 0, \\ -\sqrt{\det\, g(X_i^{\mathrm{v}}, X_j^{\mathrm{v}})} & \text{otherwise,} \end{cases}$$

$\mathrm{d}\mu_p$ is called the *(oriented) volume form* on the tangent space T_pM. Let

$$\mu_p := \iota_C\, \mathrm{d}\mu_p$$

be the induced volume form on the indicatrix hypersurface which provides an orientation for the manifold S_p. The integral of a (continuous) function f over S_p is defined as the integral of an $(n-1)$-form on an oriented manifold of dimension $n-1$ as usual:

$$\int_{S_p} f := \int_{S_p} f\mu_p.$$

Actually, the orientation was convenient but not necessary in the definition, etc. (for the citation, see [30, p. 150]). Indeed, if we change the orientation on the manifold M, then the orientation changes on the indicatrix hypersurface. For a moment, let us denote by S_p^+ and S_p^- the manifold S_p equipped with different orientations, we have that

$$\int_{S_p} f := \int_{S_p^+} f\mu_p = -\int_{S_p^-} f\mu_p = \int_{S_p^-} f(-\mu_p).$$

This means that the mapping

$$p \in M \to \int_{S_p} f$$

is well defined even if there could not be nowhere-vanishing n-form on the manifold M.

Definition 2. Let (M, E) be a positive definite Finsler manifold, the *associated Riemannian metric* is defined by the formula

$$\gamma_0(X, Y)(p) := \int_{S_p} g(X^{\mathrm{v}}, Y^{\mathrm{v}})$$

(for a similar construction, see [21]). The Lévi–Civita connection of this metric is called the *associated linear connection of the Finsler manifold*.

Proposition 1 ([27]). *The canonical (linear) connection of a positive definite Berwald manifold is just the Lévi–Civita connection of the associated Riemannian metric.*

1.5. Conformal equivalence [8] (see also [19,24])

Two Finsler manifolds (M, E) and $(M, \tilde{E})$ are said to be *conformally equivalent* if there exists a positive smooth function $\varphi : \mathcal{T}M \to \mathbb{R}$ such that

$$\tilde{g} = \varphi g.$$

The function φ is called the *scale function* or the *proportionality function*. If $\tilde{g} = \varphi g$, then

$$\tilde{E} = \tfrac{1}{2}\tilde{g}(C, C) = \tfrac{1}{2}\varphi g(C, C) = \varphi E,$$

where C is the so-called *Liouville vector field*. It is also well known due to Knebelman that the scale function between conformally equivalent Finsler manifolds is a vertical lift, i.e. φ always can be written in the form

$$\varphi = \exp \circ \alpha^{\mathrm{v}},$$

where $\alpha^{\mathrm{v}} := \alpha \circ \pi$ (see [17]). Therefore

$$\mathrm{d}\tilde{\mu}_p = \varphi(p)^{n/2}\,\mathrm{d}\mu_p. \tag{2}$$

We have the transformation formula

$$\tilde{S} = S - \alpha^{\mathrm{c}} C + E \text{ grad } \alpha^{\mathrm{v}} \tag{3}$$

and, consequently, for any vector field $X \in \mathfrak{X}(M)$

$$X^{\tilde{h}} = X^{\mathrm{h}} - \tfrac{1}{2}\alpha^{\mathrm{c}} X^{\mathrm{v}} - \tfrac{1}{2}(X\alpha)^{\mathrm{v}} C - E\mathcal{C}(F \text{ grad } \alpha^{\mathrm{v}}, X^{\mathrm{c}}) + \tfrac{1}{2} X^{\mathrm{v}} E \text{ grad } \alpha^{\mathrm{v}}. \tag{4}$$

Lemma 2. *Let (M, E) and $(M, \tilde{E})$ be conformally equivalent Finsler manifolds and suppose that the function f is homogeneous of degree k, then*

$$\int_{\tilde{S}_p} f = \varphi(p)^{-k/2} \int_{S_p} f. \tag{5}$$

Proof. Since the form $\mathrm{d}\mu_p$ has the homogeneity property

$$\mathcal{L}_C \mathrm{d}\mu_p = n\,\mathrm{d}\mu_p,$$

and, by our assumption, $\mathcal{L}_C f = kf$, the Stokes theorem shows that

$$\int_{B_p} f := \int_{B_p} f\,\mathrm{d}\mu_p = \frac{1}{n+k} \int_{B_p} \mathcal{L}_C(f\,\mathrm{d}\mu_p) = \frac{1}{n+k} \int_{B_p} \mathrm{d}\iota_C(f\,\mathrm{d}\mu_p)$$
$$= \frac{1}{n+k} \int_{S_p} \iota_C(f\,\mathrm{d}\mu_p) = \frac{1}{n+k} \int_{S_p} f\mu_p = \frac{1}{n+k} \int_{S_p} f,$$

where B_p denotes the solid indicatrix body at the point p. If

$$j : v \in T_pM \to j(v) := \frac{\tilde{L}}{L}(v)v,$$

then we have

$$\int_{\tilde{B}_p} f := \int_{\tilde{B}_p} f\,\mathrm{d}\tilde{\mu}_p = \int_{j^{-1}(B_p)} \left(\frac{\tilde{L}}{L}\right)^n \left(\frac{L}{\tilde{L}}\right)^n (f \circ j^{-1}) \circ j\,\mathrm{d}\tilde{\mu}_p$$
$$= \int_{B_p} \left(\frac{L}{\tilde{L}}\right)^n (f \circ j^{-1})\mathrm{d}\tilde{\mu}_p = \int_{B_p} \left(\frac{L}{\tilde{L}}\right)^{n+k} f\,\mathrm{d}\tilde{\mu}_p \overset{(2)}{=} \varphi(p)^{-k/2} \int_{B_p} f\,\mathrm{d}\mu_p,$$

and the formula (5) follows immediately. □

2. Conformally invariant differential forms on the underlying manifold

In what follows, we consider two conformally equivalent Finsler manifolds (M, E) and $(M, \tilde{E})$; let us denote by γ_0 and $\tilde{\gamma}_0$ their associated Riemannian metrics, respectively. It can be easily seen that the associated metrics are also conformally equivalent with a proportionality function such that its vertical lift $\varphi = \exp \circ \alpha^{\mathrm{v}}$ is just the original scale function (between the Finsler manifolds). Using the transformation formula (4) for the Riemannian manifolds (M, γ_0) and $(M, \tilde{\gamma}_0)$ we have that

$$\tilde{h}_0 = h_0 - \tfrac{1}{2}\alpha^{\mathrm{c}} J - \tfrac{1}{2}\mathrm{d}\alpha^{\mathrm{v}} \otimes C + \tfrac{1}{2}\mathrm{d}_J E_0 \otimes \mathrm{grad}_0^{\mathrm{v}}\, \alpha, \tag{6}$$

where $\tilde{h}_0$ and h_0 denotes the induced horizontal structures by the Lévi–Civita connections; the vector field

$$\mathrm{grad}_0^{\mathrm{v}}\, \alpha := (\mathrm{grad}_0\, \alpha)^{\mathrm{v}}$$

is just the vertical lift of the usual gradient with respect to the Riemannian metric γ_0. On the other hand, the associated sprays $\tilde{S}_0$ and S_0 are related as follows:

$$\tilde{S}_0 = S_0 - \alpha^{\mathrm{c}} C + E_0\, \mathrm{grad}_0^{\mathrm{v}}\, \alpha. \tag{7}$$

It is well known that the derivative of the vertical lift α^{v} with respect to an arbitrary semispray S gives the complete lift of the function α. This means that

$$\frac{\tilde{S}_0 \tilde{E}}{\tilde{E}} = \alpha^{\mathrm{c}} + \frac{\tilde{S}_0 E}{E} \overset{(7)}{=} \frac{S_0 E}{E} - \alpha^{\mathrm{c}} + \frac{E_0}{E}(\mathrm{grad}_0^{\mathrm{v}}\, \alpha)E, \tag{8}$$

and, consequently,

$$\frac{\tilde{S}_0 \tilde{E}}{\tilde{E}} - \frac{S_0 E}{E} + \alpha^{\mathrm{c}} = \frac{E_0}{E}(\mathrm{grad}_0^{\mathrm{v}}\, \alpha)E. \tag{9}$$

Since for *any* horizontal endomorphism h

$$\mathrm{d}_{\mathrm{h}}\alpha^{\mathrm{v}} = \mathrm{d}\alpha^{\mathrm{v}},$$

and the tensor field $\frac{\mathrm{d}_J E_0}{E_0}$ is conformally invariant, the formula

$$\frac{\mathrm{d}_{\tilde{h}_0}\tilde{E}}{\tilde{E}} - \frac{1}{2}\frac{\tilde{S}_0\tilde{E}}{\tilde{E}}\frac{\mathrm{d}_J\tilde{E}_0}{\tilde{E}_0} = \frac{\mathrm{d}_{h_0}E}{E} - \frac{1}{2}\frac{S_0 E}{E}\frac{\mathrm{d}_J E_0}{E_0} + \frac{1}{2}\alpha^{\mathrm{c}}\left(\frac{\mathrm{d}_J E_0}{E_0} - \frac{\mathrm{d}_J E}{E}\right) \tag{10}$$

is a direct consequence of Eqs. (6) and (9). We put

$$\rho := \frac{\mathrm{d}_{h_0}E}{E} - \frac{1}{2}\frac{S_0 E}{E}\frac{\mathrm{d}_J E_0}{E_0}, \qquad f := \ln\frac{E_0}{E} \tag{11}$$

and the implication

$$\tilde{\rho} = \rho + \frac{1}{2}\alpha^{\mathrm{c}}\,\mathrm{d}_J f \Rightarrow \mathrm{d}_J\tilde{\rho} = \mathrm{d}_J\rho + \frac{1}{2}\mathrm{d}\alpha^{\mathrm{v}} \wedge \mathrm{d}_J f \tag{12}$$

follows immediately. Consider the vector field

$$\Theta := -E \text{ grad } f$$

satisfying the relation

$$\iota_\Theta \omega = -E\,\mathrm{d}f,$$

an easy calculation shows that

$$\iota_{J\Theta}\omega = E\,\mathrm{d}_J f \Rightarrow \|J\Theta\|^2 = E\,\mathrm{d}_J f(\Theta), \tag{13}$$

where the norm is taken with respect to the Riemann–Finsler metric g.

Lemma 3. *The vector field $J\Theta$ is conformally invariant.*

Proof. For any vector field $X \in \mathfrak{X}(TM)$

$$\tilde{E}\,\mathrm{d}_J \tilde{f}(X) \overset{(13)}{=} \tilde{\omega}(J\tilde{\Theta}, X) = \tilde{g}(J\tilde{\Theta}, JX) = \varphi g(J\tilde{\Theta}, JX).$$

On the other hand, $\mathrm{d}_J \tilde{f} = \mathrm{d}_J f$ and, consequently,

$$\tilde{E}\,\mathrm{d}_J \tilde{f}(X) = \varphi(E\,\mathrm{d}_J f(X)) \overset{(13)}{=} \varphi\omega(J\Theta, X) = \varphi g(J\Theta, JX),$$

which implies that $J\tilde{\Theta} = J\Theta$ as was to be stated. □

Lemma 4. *Keeping our previous notations it follows that*

$$\iota_{\tilde{\Theta}}\,\mathrm{d}_J\tilde{\rho} - \frac{1}{2}\frac{\tilde{S}\tilde{E}_0}{\tilde{E}_0}\,\mathrm{d}_J\tilde{f} = \iota_\Theta\,\mathrm{d}_J\rho - \frac{1}{2}\frac{SE_0}{E_0}\,\mathrm{d}_J f - \frac{1}{2E}\,\|J\Theta\|^2\ \mathrm{d}\alpha^{\mathrm{v}}. \tag{14}$$

Proof. Lemma 3 shows that the difference vector field $\tilde{\Theta} - \Theta$ is vertical, i.e. $\iota_{\tilde{\Theta}-\Theta}\,\mathrm{d}_J\tilde{\rho} = 0$ and, consequently,

$$\iota_{\tilde{\Theta}}\,\mathrm{d}_J\tilde{\rho} = \iota_\Theta\,\mathrm{d}_J\tilde{\rho} = \iota_\Theta\,\mathrm{d}_J\rho + \tfrac{1}{2}\mathrm{d}\alpha^{\mathrm{v}}(\Theta)\,\mathrm{d}_J f - \tfrac{1}{2}\mathrm{d}_J f(\Theta)\,\mathrm{d}\alpha^{\mathrm{v}}, \tag{15}$$

where, as we have seen above, $\|J\Theta\|^2 = E\,\mathrm{d}_J f(\Theta)$. Since

$$\mathrm{d}\alpha^{\mathrm{v}}(\Theta) = \omega(\text{grad } \alpha^{\mathrm{v}}, \Theta) = -\omega(\Theta, \text{grad } \alpha^{\mathrm{v}}) = E \text{ grad } \alpha^{\mathrm{v}}(f) \overset{(3)}{=} (\tilde{S} - S)f,$$

we have Eq. (14) immediately. □

Definition 3. Let us define the function $\sigma : M \to \mathbb{R}$ as follows:

$$\sigma(p) := \int_{S_p} \frac{1}{2E}\,\|J\Theta\|^2\,.$$

Lemma 5. *Let* (M, E) *be a generalized Berwald manifold, then the function* σ *is strictly positive or, it is identically zero and the manifold is Riemannian.*

Proof. Suppose that $\sigma(p) = 0$, then $J\Theta$ vanishes on the tangent space T_pM. This means that there exists a (positive) constant k such that

$$E_{|T_pM} = \frac{1}{k}E_{0|T_pM},$$

and the function $E_{|T_pM}$ is quadratic. Since (M, E) is a generalized Berwald manifold we have a linear connection ∇ on the underlying manifold such that the indicatrix hypersurfaces are invariant under the parallel transport, i.e.

$$E_{|T_pM} = E_{|T_qM} \circ \tau,$$

where $\tau : T_pM \to T_qM$ is the linear isomorphism induced by the parallel transport with respect to ∇ along a curve joining p and q. Therefore, for *any* point $q \in M$ the function

$$E_{|T_qM} = \frac{1}{k}E_{0|T_pM} \circ \tau^{-1}$$

is quadratic provided, of course, that the manifold is connected. In other words, the Finsler manifold (M, E) reduces to a Riemannian manifold such that it is homothetic (conformal equivalence with a constant scale function) to the associated Riemannian manifold (M, γ_0) – the constant is just the area of the standard Euclidean unit sphere of dimension $n - 1$. Therefore $J\Theta = 0$ or, in an equivalent form, the function σ is identically zero as was to be stated. □

Remark 2. Note again that the Wagner manifolds form an important class of the generalized Berwald manifolds as those which are conformal to a Berwald manifold. The statement also holds in case of a locally conformally Berwald manifold all of whose points have a connected open neighbourhood U together with a function $\alpha \in C^\infty(U)$ such that the local conformal change $\tilde{g} := \varphi g$ results in a Berwald manifold; notations as usual. If the function σ vanishes at a point $p \in M$ then, using the local version of Lemma 5, it also vanishes on the neighbourhood U, i.e. σ is strictly positive or it is identically zero provided that the underlying manifold is connected. We have an interesting alternative reasoning by the help of Theorem 3 which states that the scale function between conformally equivalent Berwald manifolds must be constant unless they are Riemannian. This result has been proved in our previous paper [26] as the solution of the generalized Matsumoto's problem (see also [25]). As one of the most important consequences it follows that *although there is no global scale function in case of a locally conformally Berwald manifold, the linear Wagner connection induced by the collection of the local functions is globally well defined because it involves only the exterior derivatives.* According to this observation, the original proof of Lemma 5 works without any modification.

Definition 4. Let us define the differential 1-form $\eta : \mathfrak{X}(M) \to C^\infty(M)$ as follows:

$$\eta(X)_p := \int_{S_p} \mathrm{d}_J \rho(\Theta, X^c) - \frac{1}{2}\frac{SE_0}{E_0}\,\mathrm{d}_J f(X^c).$$

Theorem 1. *Let (M, E) be a Finsler manifold and suppose that the function σ is strictly positive, then the differential form*

$$\vartheta := \frac{1}{\sigma}\left(\mathrm{d}\eta - \frac{1}{\sigma}\mathrm{d}\sigma \wedge \eta\right)$$

is conformally invariant.

Proof. According to Lemma 2 it follows that $\tilde{\sigma} = \sigma$; on the other hand, by Lemma 4

$$\begin{aligned}\tilde{\eta}(X)_p &= \int_{\tilde{S}_p} \mathrm{d}_J \rho(\Theta, X^c) - \frac{1}{2}\frac{SE_0}{E_0}\,\mathrm{d}_J f(X^c) - \sigma(p)X_p(\alpha)\\ &=^{(5)} \int_{S_p} \mathrm{d}_J \rho(\Theta, X^c) - \frac{1}{2}\frac{SE_0}{E_0}\,\mathrm{d}_J f(X^c) - \sigma(p)X_p(\alpha) = \eta(X)_p - \sigma(p)X_p(\alpha).\end{aligned}$$

This means that

$$\frac{\tilde{\eta}}{\tilde{\sigma}} = \frac{\eta}{\sigma} - \mathrm{d}\alpha, \tag{16}$$

and the exterior derivative ϑ of the form $\frac{\eta}{\sigma}$ is conformally invariant. □

Theorem 2. *Let (M, E) be a non-Riemannian Finsler manifold; it is a locally conformally Berwald manifold if and only if the horizontal endomorphism induced by the linear connection*

$$\bar{\nabla}_X Y := \nabla_X Y + \frac{1}{2\sigma}(\eta(Y)X - \gamma_0(X, Y)\eta^\sharp)$$

is conservative, i.e. $d_{\bar{h}}E = 0$ and $\vartheta = 0$; the sharp operator is taken with respect to the associated Riemannian metric γ_0 and ∇ denotes its Lévi–Civita connection.

Proof. Suppose that (M, E) is a locally conformally Berwald manifold, then for any point $p \in M$ there exists a neighbourhood U together with a function $\alpha \in C^\infty(U)$ such that the local conformal change $\tilde{g} = \varphi g$ results in a Berwald manifold. According to Lemma 5 the function σ is strictly positive and the Eq. (16) reduces to the form

$$\frac{\eta}{\sigma} = \mathrm{d}\alpha, \tag{17}$$

therefore the vanishing of the exterior derivative ϑ is clear. As it is well known from the general theory of Wagner connections and Wagner manifolds, the Wagner connection induced by the function $-\frac{1}{2}\alpha$ is linear. Explicitly, the Wagner endomorphism $\bar{h}$ and the canonical horizontal endomorphism $\tilde{h}$ of the resulting Berwald manifold are related as follows:

$$\bar{h} = \tilde{h} + \tfrac{1}{2}\mathrm{d}\alpha^{\mathrm{v}} \otimes C$$

(for details, see [22,23]). On the other hand, as Proposition 1 states, the covariant derivative operator induced by $\tilde{h}$ must be the Lévi-Civita connection of the associated Riemannian metric $\tilde{\gamma}_0$. This means that

$$\bar{h} = \tilde{h}_0 + \tfrac{1}{2}\mathrm{d}\alpha^{\mathrm{v}} \otimes C \overset{(6)}{=} h_0 - \tfrac{1}{2}\alpha^{\mathrm{c}} J + \tfrac{1}{2}\mathrm{d}_J E_0 \otimes \operatorname{grad}_0^{\mathrm{v}} \alpha,$$

which is just the horizontal endomorphism induced by the linear connection $\bar{\nabla}$ and, as a Wagner endomorphism, it is conservative.

Conversely, the vanishing of ϑ implies (at least locally) the existence of a function α such that

$$\frac{\eta}{\sigma} = \mathrm{d}\alpha. \tag{18}$$

For the sake of simplicity we omit the local neighbourhood in our notations. The conformal change with the scale function $\varphi := \exp \circ \alpha^{\mathrm{v}}$ results in a Finsler manifold $(M, \tilde{E})$ such that

$$\tilde{h}_0 = \bar{h} - \frac{1}{2}\mathrm{d}\alpha^{\mathrm{v}} \otimes C,$$

where $\bar{h}$ denotes the horizontal endomorphism induced by the linear connection $\bar{\nabla}$. This means that

$$\mathrm{d}_{\tilde{h}_0}\tilde{E} = \mathrm{d}_{\bar{h}}\tilde{E} - \tilde{E}\,\mathrm{d}\alpha^{\mathrm{v}} = \varphi\,\mathrm{d}_{\bar{h}}E,$$

where the right-hand side vanishes by the condition of conservativity. Therefore the Lévi–Civita connection of the associated Riemannian metric $\tilde{\gamma}_0$ must be the canonical connection of the Finsler manifold $(M, \tilde{E})$ and, consequently, it is a Berwald manifold. □

Theorem 3. *The scale function between conformally equivalent Berwald manifolds must be constant unless the manifolds are Riemannian.*

Proof. It is a direct consequence of Eq. (16) which reduces to the final simple form

$$\mathrm{d}\alpha = 0$$

in case of such a conformal change. □

Exercise. Using the formula (16) prove that the connection

$$\bar{\bar{\nabla}}_X Y := \nabla_X Y + \frac{1}{2\sigma}(\eta(X)Y + \eta(Y)X - \gamma(X, Y)\eta^{\sharp})$$

is conformally invariant.

Remark 3. We should note that the previous construction of a conformally invariant differential form on the underlying manifold is far from being unique. There are several ways for the modification.

(i) In case of special Finsler manifolds such as Randers manifolds, the Riemannian part of the starting "data" seems to be the best *associated Riemannian metric* although, as we have seen above, the integral formulas provides such kind of (in general different) geometrical structures. As an application of our results we are going to prove that the Riemannian part has all of good properties to be an associated Riemannian structure in the sense of the following remark.

(ii) According to the nature of the conformality problem there are two important requirements for the association of geometrical structures. First of all the associated Riemannian metrics must be conformally equivalent in case of conformally equivalent Finsler manifolds with a common scale function. On the other hand, the associated metric is required to satisfy the Riemann-metrizability condition in case of a Berwald manifold which means that the canonical linear connection of the Finsler manifold and the Lévi–Civita connection coincide.

(iii) Technically we can prefer the associated Riemannian metric over the Riemann–Finsler metric of the Finsler manifold in order to construct the gradient vector field of the function $f := \ln \frac{E_0}{E}$ by the formula

$$\iota_{\Theta_0}\omega_0 = -E_0\,\mathrm{d}f,$$

where ω_0 is the fundamental form of the Finsler (esp. the Riemannian) manifold (M, E_0). Instead of Eq. (14) it follows that

$$\iota_{\tilde{\Theta}_0}\,\mathrm{d}_J\tilde{\rho} + \frac{1}{2}\frac{\tilde{S}_0\tilde{E}}{\tilde{E}}\,\mathrm{d}_J\tilde{f} = \iota_{\Theta_0}\,\mathrm{d}_J\rho + \frac{1}{2}\frac{S_0E}{E}\,\mathrm{d}_J f - \frac{1}{2E_0}\,\|J\Theta_0\|^2\,d\alpha^{\mathrm{v}}, \tag{19}$$

the norm is taken with respect to the Riemann–Finsler metric g_0 – it is just the vertical lift of the associated Riemannian metric.

(iv) As the next step we can integrate on the Riemannian indicatrices with respect to their own oriented volume forms to introduce a function σ_0 and a differential form η_0 such that the exterior derivative of the scale function can be expressed as a difference

$$\mathrm{d}\alpha = \frac{\eta_0}{\sigma_0} - \frac{\tilde{\eta}_0}{\tilde{\sigma}_0},$$

it is actually the original Kikuchi's "excellent idea" (see [16]). The only requirement for the integration is to satisfy the relation (5).

(v) We note that in case of a Wagner manifold all of these modifications give the same final result because there is a special conformal change such that the left-hand side of both Eqs. (14) and (19) is identically zero. Explicitly

$$\frac{\eta}{\sigma} = \frac{\eta_0}{\sigma_0},$$

they are just the exterior derivative of the scale function which provides the conformal equivalence to a Berwald manifold – as we have seen above it is uniquely determined up to an additive constant.

(vi) In what follows Randers manifolds are investigated to demonstrate how we can simplify (or modify) the process in special cases.

3. Conformally Berwald Randers manifolds

Definition 5. Consider a Riemannian manifold (M, γ_*) together with a nonzero 1-form β on the underlying manifold such that

$$\sup_{v \in \mathcal{T}M} \frac{\beta(v)}{L_*(v)} < 1,$$

where

$$L_*(v) := \sqrt{\gamma_*(v, v)}$$

is the norm of the tangent vectors with respect to the metric γ_*. The *Randers manifold* constructed from the Riemannian manifold (M, γ_*) by perturbation with β is defined as a Finsler manifold (M, E), where

$$L := L_* + \beta$$

and $E := \frac{1}{2}L^2$, as usual.

Theorem 4. *Let (M, E) be a Randers manifold and suppose that $\beta_p \neq 0$, then the associated Riemannian metric can be expressed as the combination*

$$\gamma_0(X, Y) = A\gamma_*(X, Y) + B\beta(X)\beta(Y),$$

where A and B are smooth functions on a connected open neighbourhood around the point $p \in M$.

Proof. Consider a coordinate system $u^1, \ldots, u^n$ such that

(i) the coordinate vector fields form a γ_*-orthonormal system $(e_1, \ldots, e_n)$ of tangent vectors at the point p,

(ii) in terms of these local coordinates $\beta = K\, du^n$, where $K := \|\beta^\sharp\|$ and the sharp operator together with the norm is taken with respect to the Riemannian part γ_* of the initial data.

As it is well known for any indices $1 \leq i, j \leq n-1$,

$$g_{ij} = \frac{L}{L_*}\delta_{ij} - Ky_*^i y_*^j y_*^n, \qquad g_{in} = \frac{L}{L_*}\delta_{in} - Ky_*^i (y_*^n)^2 + Ky_*^i,$$

$$g_{nn} = \frac{L}{L_*}\delta_{nn} - K(y_*^n)^3 + 2Ky_*^n + K^2,$$

where $(y^1, \ldots, y^n)$ are the dual basis of $(e_1, \ldots, e_n)$ and

$$y_*^1 := \frac{y^1}{L_*}, \ldots, y_*^n := \frac{y^n}{L_*}.$$

Moreover,

$$\mathrm{d}\mu_p = \left(\frac{L}{L_*}\right)^{(n+1)/2} \mathrm{d}\mu_p^* \tag{20}$$

for the foundations of Randers manifold we can refer e.g. [1,14]. Using the diffeomorphism

$$j : v \in T_pM \to j(v) := \frac{L}{L_*}(v)v,$$

it follows that

$$\begin{aligned}\int_{B_p} f := \int_{B_p} f\,\mathrm{d}\mu_p &= \int_{j^{-1}(B_p^*)} \left(\frac{L}{L_*}\right)^n \left(\frac{L_*}{L}\right)^n (f \circ j^{-1}) \circ j\,\mathrm{d}\mu_p \\ &= \int_{B_p^*} \left(\frac{L_*}{L}\right)^n (f \circ j^{-1})\,\mathrm{d}\mu_p \overset{(20)}{=} \int_{B_p^*} \left(\frac{L_*}{L}\right)^{(n-1)/2} (f \circ j^{-1})\,\mathrm{d}\mu_p^* \\ &= \int_{B_p^*} \left(\frac{L_*}{L}\right)^{k+(n-1)/2} f\,\mathrm{d}\mu_p^*,\end{aligned}$$

and, consequently,

$$\int_{S_p} f = \int_{S_p^*} \left(\frac{L_*}{L}\right)^{k+(n-1)/2} f \tag{21}$$

provided that the function f is homogeneous of degree k. In case of dimension 2 we have

$$\gamma_0(e_1, e_1) \overset{(i)}{=} \gamma_*(e_1, e_1) \int_0^{2\pi} \sqrt{1 + K_p \sin v}\left(1 - \frac{K_p}{1 + K_p \sin v}\cos^2 v \sin v\right) \mathrm{d}v,$$

$$\gamma_0(e_1, e_2) \overset{(i)}{=} \int_0^{2\pi} \frac{K_p}{\sqrt{1 + K_p \sin v}} \cos^3 v\,\mathrm{d}v = 0,$$

$$\gamma_0(e_2, e_2) \overset{(i)}{=} \gamma_*(e_2, e_2) \int_0^{2\pi} \sqrt{1 + K_p \sin v}\left(1 - \frac{K_p}{1 + K_p \sin v}\cos^2 v \sin v\right) \mathrm{d}v$$

$$+K_p^2\int_0^{2\pi}\frac{1}{K_p\sqrt{1+K_p\sin v}}(1+2\cos^2 v)\sin v+\frac{1}{\sqrt{1+K_p\sin v}}\,\mathrm{d}v,$$

where $K_p := K(p)$. If

$$A(p):=\int_0^{2\pi}\sqrt{1+K_p\sin v}\left(1-\frac{K_p}{1+K_p\sin v}\cos^2 v\sin v\right)\mathrm{d}v,$$

$$B(p):=\int_0^{2\pi}\frac{1}{K_p\sqrt{1+K_p\sin v}}(1+2\cos^2 v\sin v)+\frac{1}{\sqrt{1+K_p\sin v}}\,\mathrm{d}v,$$

then the decomposition

$$\gamma_0(X,Y)=A\gamma_*(X,Y)+B\beta(X)\beta(Y)$$

follows immediately. In case of higher-dimensional manifolds the reasoning is essentially the same using the parameterization

$$r:(u,v)\in H\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to r(u,v):=(\rho(u)\cos v,\ \sin v)\in S_p^*,$$

where $\rho : H\to s_p^*$ is the parameterization of the $(n-2)$-dimensional unit sphere

$$s_p^* := \operatorname{Ker}\beta_p\cap S_p^*.$$

Since $y_*^1,\ldots,y_*^{n-1}$ are surface harmonics of degree 1 constituting an orthogonal system as the elements of the pre-Hilbert space of continuous functions on the sphere s_p^* with the usual scalar product

$$\langle f,g\rangle := \int_{s_p^*} fg,$$

it follows that

$$\int_{s_p^*} y_*^1=\cdots=\int_{s_p^*} y_*^{n-1}=0,$$

and

$$\int_{s_p^*}(y_*^1)^2=\cdots=\int_{s_p^*}(y_*^{n-1})^2=\frac{a}{n-1},$$

where a denotes the area of s_p^* or, in an equivalent way, the area of the standard Euclidean unit sphere of dimension $(n-2)$. Therefore, as an easy calculation shows

$$\gamma_0(e_i,e_j)=0$$

provided that $i \neq j$. On the other hand

$$\gamma_0(e_i, e_i) = \gamma_*(e_i, e_i) \int_{-\pi/2}^{\pi/2} \frac{a}{(1 + K_p \sin v)^{(n-3)/2}} \cos^{n-2} v \, \mathrm{d}v - \frac{K_p}{n-1} \int_{-\pi/2}^{\pi/2} \frac{a}{(1 + K_p \sin v)^{(n-1)/2}} \cos^n v \sin v \, \mathrm{d}v,$$

$$\gamma_0(e_n, e_n) = \gamma_*(e_n, e_n) \int_{-\pi/2}^{\pi/2} \frac{a}{(1 + K_p \sin v)^{(n-3)/2}} \cos^{n-2} v \, \mathrm{d}v - \frac{K_p}{n-1} \int_{-\pi/2}^{\pi/2} \frac{a}{(1 + K_p \sin v)^{(n-1)/2}} \cos^n v \sin v \, \mathrm{d}v + K_p^2 \int_{-\pi/2}^{\pi/2} \frac{a}{K_p(1 + K_p \sin v)^{(n-1)/2}} \left(1 + \frac{n}{n-1} \cos^2 v\right) \sin v \cos^{n-2} v \mathrm{d}v + K_p^2 \int_{-\pi/2}^{\pi/2} \frac{a}{(1 + K_p \sin v)^{(n-1)/2}} \cos^{n-2} v \, \mathrm{d}v.$$

Note that the second term in the expression of $\gamma_0(e_n, e_n)$ is the consequence of a simple supplement. If

$$A(p) := \int_{-\pi/2}^{\pi/2} \frac{a}{(1 + K_p \sin v)^{(n-3)/2}} \cos^{n-2} v \, \mathrm{d}v - \frac{K_p}{n-1} \int_{-\pi/2}^{\pi/2} \frac{a}{(1 + K_p \sin v)^{(n-1)/2}} \cos^n v \sin v \, \mathrm{d}v,$$

$$B(p) := \int_{-\pi/2}^{\pi/2} \frac{a}{K_p(1 + K_p \sin v)^{(n-1)/2}} \left(1 + \frac{n}{n-1} \cos^2 v\right) \sin v \cos^{n-2} v \, \mathrm{d}v + \int_{-\pi/2}^{\pi/2} \frac{a}{(1 + K_p \sin v)^{(n-1)/2}} \cos^{n-2} v \, \mathrm{d}v,$$

then the decomposition

$$\gamma_0(X, Y) = A\gamma_*(X, Y) + B\beta(X)\beta(Y)$$

follows immediately. □

Remark 4. As an application of the decomposition formula, we give a new proof of the following well-known theorem due to Matsumoto which states that the Riemannian part of the initial data can be considered as an associated Riemannian metric in the sense of Remark 3 (ii) (see also Remark 3 (i)).

Theorem 5. *Let (M, E) be a Randers manifold and suppose that it is a Berwald manifold, then the canonical (linear) connection and the Lévi–Civita connection of the metric γ_* coincide.*

Proof. It is enough to prove the statement over the set of "regular" points $p \in M$, where $\beta_p \neq 0$. Since (M, E) is a Berwald manifold the canonical horizontal endomorphism h arises from a linear connection on the underlying manifold. It is conservative, i.e. the h-covariant derivatives of the energy function or, in an equivalent way, the fundamental function L vanish which implies that

$$d_h L_* = -d_h \beta.$$

Therefore *the form* $d_h L_*$ *is linear in its second n variables* $y^1, \ldots, y^n$. Proposition 1 states that $h = h_0$, i.e. the Barthel endomorphism is induced by the Lévi–Civita connection of the associated metric γ_0. Using the previous theorem, the decomposition formula

$$E_0 = A^v E_* + \tfrac{1}{2} B^v \beta^2$$

follows immediately. Since $d_h E_0 = 0$,

$$L_* d_h L_* = -\frac{1}{A^v} \left(E_* \, dA^v + \frac{1}{2} \beta^2 \, dB^v + B^v \beta \, d_h \beta \right)$$

and we have that *the form* $L_* \, d_h L_*$ *is quadratic in its second n variables* $y^1, \ldots, y^n$. The emphasized observations implies that $d_h L_* = 0$ and, consequently, $h_* = h (= h_0)$ as was to be stated in terms of the associated linear connections. □

Remark 5. As a direct consequence of the previous theorem, we have that the Randers manifold (M, E) is a Berwald manifold if and only if β is parallel with respect to the metric γ_*.

Lemma 6. *Let* (M, E) *be a Randers manifold and suppose that it is a generalized Berwald manifold, then the function* $\sigma_* := \frac{1}{2} K^2$ *is strictly positive or, it is identically zero and the manifold is Riemannian.*

Proof. Since (M, E) is a generalized Berwald manifold, we have a linear connection ∇ on the underlying manifold such that the indicatrix hypersurfaces are invariant under the parallel transport. In other words, the induced horizontal endomorphism h is conservative, i.e. the h-covariant derivatives of the energy function or, in an equivalent way, the fundamental function L vanish. The main result of our previous paper [27] states that this linear connection must be metrical with respect to the associated metrical γ_0. In a similar way as in the proof of Theorem 5 it follows that $d_h L_* = 0$ and, consequently, the form β is parallel with respect to the connection ∇. This means that the existence of a single zero implies the function σ_* to be identically zero. □

Definition 6. We put

$$\eta_* := \nabla_{\beta^\sharp} \beta - \frac{\operatorname{div} \beta^\sharp}{n-1} \beta,$$

where the sharp operator together with the divergence is taken with respect to the Riemannian part γ_* of the initial data and ∇ denotes its Lévi–Civita connection.

Theorem 6. *Let (M, E) be a Randers manifold and suppose that the function σ_* is strictly positive, then the differential form*

$$\vartheta_* := \frac{1}{\sigma_*}\left(\mathrm{d}\eta_* - \frac{1}{\sigma_*}\mathrm{d}\sigma_* \wedge \eta_*\right)$$

is conformally invariant.

Proof. A routine calculation shows that the Lévi–Civita connections of the conformally equivalent Riemannian manifolds $(M, \tilde{\gamma}_*)$ and (M, γ_*) are related as follows:

$$\tilde{\nabla}_X Y = \nabla_X Y + \tfrac{1}{2}((X\alpha)Y + (Y\alpha)X - \gamma_*(X, Y)\,\mathrm{grad}_*\,\alpha),$$

where $\alpha := \ln\varphi$ is the logarithm of the scale function – for the sake of simplicity it is interpreted on the underlying manifold omitting the symbol of the vertical lift.

Since $\tilde{\beta} = \varphi^{1/2}\beta$ the formula

$$\varphi^{-1/2}(\tilde{\nabla}_X\tilde{\beta})(Y) = (\nabla_X\beta)(Y) - \tfrac{1}{2}(Y\alpha)\beta(X) + \tfrac{1}{2}\gamma_*(X, Y)\beta(\mathrm{grad}_*\,\alpha) \tag{22}$$

follows immediately. By the substitution $X := \beta^\sharp$

$$(\tilde{\nabla}_{\tilde{\beta}^{\tilde{\sharp}}}\tilde{\beta})(Y) = (\nabla_{\beta^\sharp}\beta)(Y) - \frac{1}{2}K^2(Y\alpha) + \frac{1}{2}\beta(Y)\beta(\mathrm{grad}_*\,\alpha).$$

We note that

$$\tilde{\beta}^{\tilde{\sharp}} = \frac{1}{\varphi}\tilde{\beta}^\sharp = \varphi^{-1/2}\beta^\sharp, \qquad \tilde{K}^2 = \tilde{\gamma}_*(\tilde{\beta}^{\tilde{\sharp}}, \tilde{\beta}^{\tilde{\sharp}}) = \frac{1}{\varphi^2}\tilde{\gamma}_*(\tilde{\beta}^\sharp, \tilde{\beta}^\sharp) = \frac{1}{\varphi}\tilde{\gamma}_*(\beta^\sharp, \beta^\sharp) = K^2,$$

$$\begin{aligned}\tilde{\mathrm{div}}\,\tilde{\beta}^{\tilde{\sharp}} &= \tilde{\mathrm{div}}\,\varphi^{-1/2}\beta^\sharp = \varphi^{-1/2}\left(\tilde{\mathrm{div}}\,\beta^\sharp - \frac{1}{2}\beta(\mathrm{grad}_*\,\alpha)\right)\\ &= \varphi^{-1/2}\left(\mathrm{div}\,\beta^\sharp + \frac{n-1}{2}\beta(\mathrm{grad}_*\,\alpha)\right),\end{aligned} \tag{23}$$

and, consequently,

$$\begin{aligned}(\tilde{\nabla}_{\tilde{\beta}^{\tilde{\sharp}}}\tilde{\beta})(Y) &= (\nabla_{\beta^\sharp}\beta)(Y) - \frac{1}{2}K^2(Y\alpha) + \frac{1}{n-1}\left(\varphi^{1/2}\tilde{\mathrm{div}}\,\tilde{\beta}^{\tilde{\sharp}} - \mathrm{div}\,\beta^\sharp\right)\beta(Y)\\ &= (\nabla_{\beta^\sharp}\beta)(Y) - \frac{1}{2}K^2(Y\alpha) + \frac{\tilde{\mathrm{div}}\,\tilde{\beta}^{\tilde{\sharp}}}{n-1}\tilde{\beta}(Y) - \frac{\mathrm{div}\,\beta^\sharp}{n-1}\beta(Y).\end{aligned}$$

Therefore

$$\frac{1}{\tilde{K}^2}\left(\tilde{\nabla}_{\tilde{\beta}^{\tilde{\sharp}}}\tilde{\beta} - \frac{\tilde{\mathrm{div}}\,\tilde{\beta}^{\tilde{\sharp}}}{n-1}\tilde{\beta}\right) = \frac{1}{K^2}\left(\nabla_{\beta^\sharp}\beta - \frac{\mathrm{div}\ \beta}{n-1}\beta\right) - \frac{1}{2}\mathrm{d}\alpha,$$

i.e.

$$\frac{\tilde{\eta}_*}{\tilde{\sigma}_*} = \frac{\eta_*}{\sigma_*} - \mathrm{d}\alpha, \tag{24}$$

and the exterior derivative ϑ_* of the form $\frac{\eta_*}{\sigma_*}$ is conformally invariant. □

Theorem 7. *Let (M, E) be a non-Riemannian Randers manifold, it is a locally conformally Berwald manifold if and only if the horizontal endomorphism induced by the linear connection*

$$\bar{\nabla}_X Y := \nabla_X Y + \frac{1}{2\sigma_*}(\eta_*(Y)X - \gamma_*(X, Y)\eta_*^\sharp)$$

is conservative, i.e. $\mathrm{d}_{\bar{h}}E = 0$ *and* $\vartheta_* = 0$, *the sharp operator is taken with respect to the Riemannian part* γ_* *of the initial data and* ∇ *denotes its Lévi–Civita connection.*

Proof. The reasoning is the same as in the proof of Theorem 2. □

Exercise. Using the formula (24) prove that the connection

$$\bar{\bar{\nabla}}_X Y := \nabla_X Y + \frac{1}{2\sigma_*}(\eta_*(X)Y + \eta_*(Y)X - \gamma_*(X, Y)\eta_*^\sharp)$$

is conformally invariant.

Remark 6. It can be easily seen from the relation (22) that

$$(\nabla_X\beta)(Y) = \tfrac{1}{2}(Y\alpha)\beta(X) - \tfrac{1}{2}\gamma_*(X, Y)\beta(\mathrm{grad}_*\ \alpha) \tag{25}$$

is a necessary and sufficient condition for a Randers manifold to be conformal to a Berwald manifold. If $X_1, \ldots, X_n$ is a local orthonormal frame, then

$$\mathrm{div}\ \beta^\sharp = \sum_{i=1}^{n}(\nabla_{X_i}\beta)(X_i) = -\frac{n-1}{2}\beta(\mathrm{grad}_*\ \alpha), \tag{26}$$

and, consequently,

$$(\nabla_X\beta)(Y) = \frac{1}{2}(Y\alpha)\beta(X) + \frac{\mathrm{div}\ \beta^\sharp}{n-1}\gamma(X, Y).$$

In case of $X := \beta^\sharp$ it follows that

$$\frac{1}{2}(Y\alpha) = \frac{1}{2\sigma_*}\eta_*(Y).$$

Therefore

$$(\nabla_X \beta)(Y) = \frac{1}{2\sigma_*}\beta(X)\eta_*(Y) + \frac{\operatorname{div}\,\beta^\sharp}{n-1}\gamma_*(X, Y), \tag{27}$$

and we have that *a non-Riemannian Randers manifold is a locally conformally Berwald manifold if and only if*

$$\nabla\beta = \frac{1}{2\sigma_*}\beta \otimes \eta_* + \frac{\operatorname{div}\,\beta^\sharp}{n-1}\gamma_*,$$

and $\vartheta_* = 0$. According to the formula (27)

$$\operatorname{div}\,\beta^\sharp = \frac{1}{2\sigma_*}\eta_*(\beta^\sharp) + \frac{n}{n-1}\operatorname{div}\,\beta^\sharp = \frac{1}{2\sigma_*}(\nabla_{\beta^\sharp}\beta)(\beta^\sharp) + \operatorname{div}\,\beta^\sharp,$$

therefore $(\nabla_{\beta^\sharp}\beta)(\beta^\sharp) = 0$. By the substitution $Y := \beta^\sharp$ into (27)

$$(\nabla_X\beta)(\beta^\sharp) = 0,$$

and the implication $\nabla K = 0$ follows immediately. This means that the vector field $\beta^\sharp$ has constant length with respect to the metric γ_*.

Exercise. Prove the “existence theorem of Wagner spaces” (sic) as a simple special case of the relation (25) (for the theorem, see [2, Theorem 7].

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